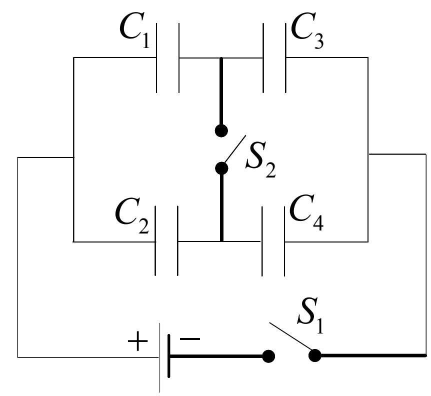
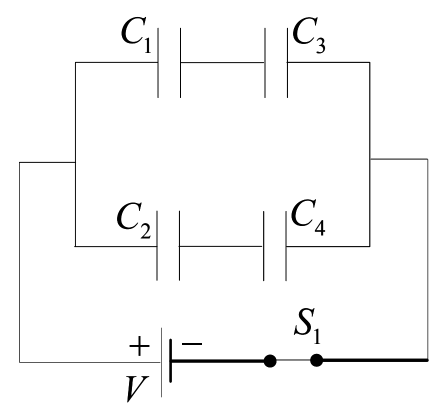


Problem: Four capacitors $C_1 = 1.00\ \mu F$, $C_2 = 2.00\ \mu F$, $C_3 = 4.00\ \mu F$, $C_4 = 4.00\ \mu F$ and a 12.0 V battery are connected as shown in the figure. If only switch 1 is closed, what is the charge on (a) capacitor 1 (b) capacitor 2, (c) capacitor 3, and (d) capacitor 4? If both switches are closed, what is the charge on (e) capacitor 1 (f) capacitor 2, (g) capacitor 3, and (h) capacitor 4?

Solution



In the case when only the switch S_1 is closed, we have the following circuit. It is easy to see that the potential difference applied to the combinations of capacitors C_1 and C_3 (connected in series); and C_2 and C_4 (also connected in series) is the same and equal to V. Due to the fact the these capacitors are connected in series they have the same amount of charge on their plates, that is

$$q_1 = q_3, \qquad \text{and} \qquad q_2 = q_4.$$

The values of charge are given by the formulas

$$q_1 = q_3 = C_{13}V, \qquad \text{and} \qquad q_2 = q_4 = C_{24}V,$$

where

$$\frac{1}{C_{13}} = \frac{1}{C_1} + \frac{1}{C_3}, \qquad \text{and} \qquad \frac{1}{C_{24}} = \frac{1}{C_2} + \frac{1}{C_4}.$$

Using,

$$C_{13} = \frac{C_1 C_3}{C_1 + C_3}, \qquad \text{and} \qquad C_{24} = \frac{C_2 C_4}{C_2 + C_4},$$

one can get

$$q_1 = q_3 = \frac{C_1 C_3}{C_1 + C_3}V, \qquad \text{and} \qquad q_2 = q_4 = \frac{C_2 C_4}{C_2 + C_4}V.$$

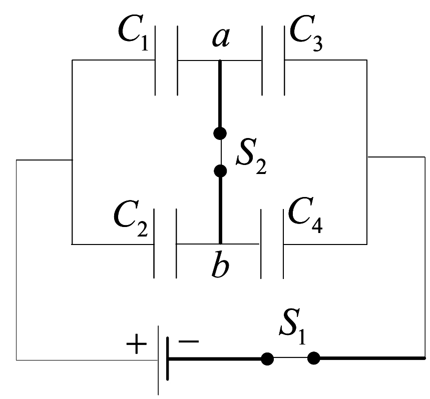
$$q_1 = q_3 = \frac{(1.00\,\mu F)(3.00\,\mu F)}{(1.00\,\mu F + 3.00\,\mu F)}(12.0V) = \frac{3.00}{4.00}10^{-6}F(12.0V) = 9.00\cdot 10^{-6}\,F\cdot V = 9.00\cdot 10^{-6}\,C,$$

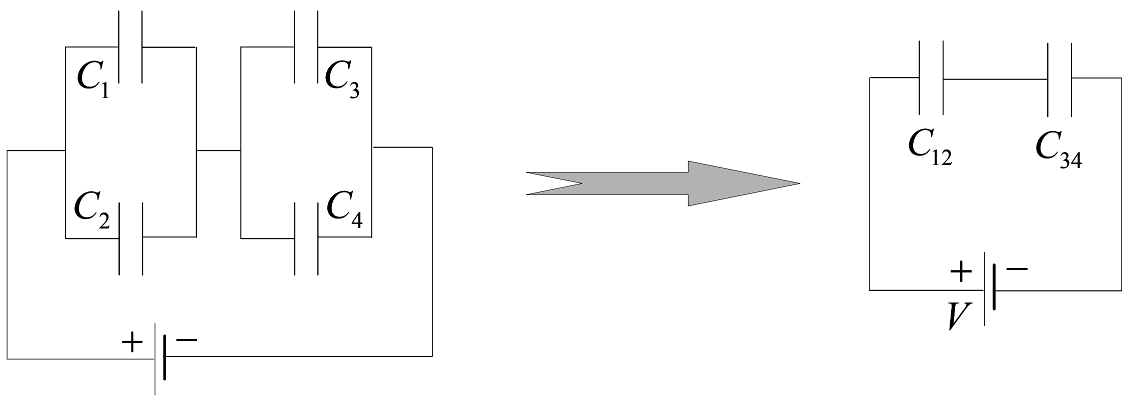
$$q_1 = q_3 = 9.00\cdot 10^{-6}\,C.$$

$$q_2 = q_4 = \frac{(2.00\,\mu F)(4.00\,\mu F)}{(2.00\,\mu F + 4.00\,\mu F)}(12.0V) = \frac{8.00}{6.00}10^{-6}F(12.0V) = 16.0\cdot 10^{-6}F\cdot V,$$

$$q_2 = q_4 = 16.0\cdot 10^{-6}C.$$

In the case when both switches are closed, we have the following circuit, in which the potential difference between the points a and b is zero. Thus, this circuit can be represented as the following equivalent circuit

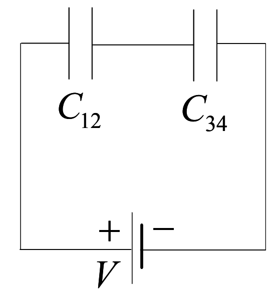




$$C_{12} = C_1 + C_2,$$

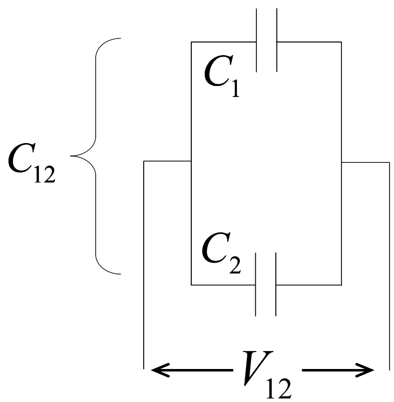
$$C_{34} = C_3 + C_4,$$

$$\frac{1}{C_{tot}} = \frac{1}{C_{12}} + \frac{1}{C_{34}}, \qquad C_{tot} = \frac{C_{12}C_{34}}{C_{12} + C_{34}}.$$



$$C_{tot} = \frac{(C_1 + C_2)(C_3 + C_4)}{C_1 + C_2 + C_3 + C_4}.$$

Both blocks of capacitors C_{13} and C_{24} have equal charge stored in them

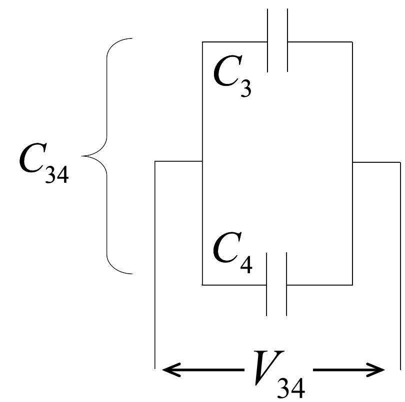


$$q_{12} = q_{34} = C_{tot}V.$$

$$q_{12} = q_1 + q_2, \qquad q_{34} = q_3 + q_4$$

$$V_{13} + V_{24} = V.$$

$$V_{12} = \frac{q_{12}}{C_{12}}, \qquad V_{34} = \frac{q_{34}}{C_{34}},$$



$$V_{12} = \frac{C_{tot}}{C_{12}}V = \frac{\cancel{(C_1 + C_2)}(C_3 + C_4)}{(C_1 + C_2 + C_3 + C_4)\cancel{(C_1 + C_2)}}V = \frac{(C_3 + C_4)}{(C_1 + C_2 + C_3 + C_4)}V,$$

$$V_{12} = \frac{(C_3 + C_4)}{(C_1 + C_2 + C_3 + C_4)}V.$$

$$V_{34} = \frac{C_{tot}}{C_{34}} V = \frac{(C_1 + C_2)\cancel{(C_3 + C_4)}}{(C_1 + C_2 + C_3 + C_4)\cancel{(C_3 + C_4)}} V = \frac{(C_1 + C_2)}{(C_1 + C_2 + C_3 + C_4)} V,$$

$$V_{34} = \frac{(C_1 + C_2)}{(C_1 + C_2 + C_3 + C_4)} V.$$

Taking into account that capacitors C_1 and C_3 are connected in parallel, that is the potential difference V_{13} is applied across both capacitors

$$q_1 = V_{12} C_1 = \frac{(C_3 + C_4) C_1}{(C_1 + C_2 + C_3 + C_4)} V, \qquad q_2 = V_{12} C_2 = \frac{(C_3 + C_4) C_2}{(C_1 + C_2 + C_3 + C_4)} V,$$

$$q_1 = \frac{(C_3 + C_4) C_1}{(C_1 + C_2 + C_3 + C_4)} V, \qquad q_2 = \frac{(C_3 + C_4) C_2}{(C_1 + C_2 + C_3 + C_4)} V.$$

Similar consideration for capacitors C_2 and C_4 gives

$$q_3 = V_{34} C_3 = \frac{(C_1 + C_2) C_3}{(C_1 + C_2 + C_3 + C_4)} V, \qquad q_4 = V_{34} C_4 = \frac{(C_1 + C_2) C_4}{(C_1 + C_2 + C_3 + C_4)} V,$$

$$q_3 = \frac{(C_1 + C_2)C_3}{(C_1 + C_2 + C_3 + C_4)}V, \qquad q_4 = \frac{(C_1 + C_2)C_4}{(C_1 + C_2 + C_3 + C_4)}V.$$

$$q_1 = \frac{(3.00\,\mu F + 4.00\,\mu F)(1.00\,\mu F)}{(10.0\mu F)}(12.0V) = \left(\frac{7.00}{10.0}\cdot 10^{-6}F\right)(12.0V) = 8.4\cdot 10^{-6}C,$$

$$q_1 = 8.4\cdot 10^{-6}C.$$

$$q_2 = \frac{(7.00\mu F)(2.00\mu F)}{(10.0\mu F)}(12.0V) = \left(\frac{14.0}{10.0}10^{-6}F\right)(12.0V) = 16.8\cdot 10^{-6}C,$$

$$q_2 = 16.8\cdot 10^{-6}C.$$

$$q_3 = \frac{(3.00\mu F)(3.00\mu F)}{(10.0\mu F)}(12.0V) = 10.8\cdot 10^{-6}C,$$

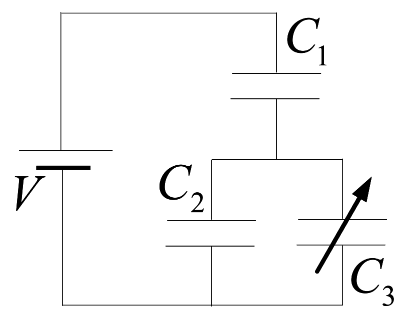
$$q_3 = 10.8\cdot 10^{-6}C.$$

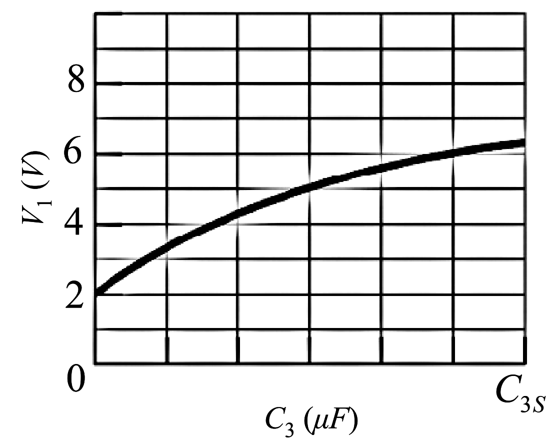
$$q_4 = \frac{(3.00\mu F)(4.00\mu F)}{(10.0\mu F)}(12.0V) = 14.4 \cdot 10^{-6} C,$$

$$q_4 = 14.4 \cdot 10^{-6} C.$$

***Problem*:** Capacitors C_1, C_2, and a variable capacitor C_3 are connected as shown in the figure. The battery supplies a potential difference V to the circuit. The dependence of the potential difference V_1 across the capacitor C_1 as a function of changing capacitance C_3 is presented in the graph.

The horizontal scale in the graph is set by C_{3S} = 12.0 μF. The potential difference V_1 approaches asymptotic value of 10 V as $C_{3S} \rightarrow \infty$. Find (a) the electric potential across the battery, (b) C_1, and (c) C_2.





Solution

The total capacitance of the circuit can be easily found because capacitors C_2 and C_3 are connected in parallel and the block C_{23} is connected in series with C_1.

$$\frac{1}{C_{123}} = \frac{1}{C_1} + \frac{1}{C_2 + C_3} = \frac{C_1 + C_2 + C_3}{C_1(C_2 + C_3)}, \qquad C_{123} = \frac{C_1(C_2 + C_3)}{(C_1 + C_2 + C_3)}$$

The total charge delivered to the circuit is equal to

$$q = C_{123}V = \frac{C_1(C_2 + C_3)}{(C_1 + C_2 + C_3)}V.$$

By the property of the series connection of capacitors we can conclude that

$$q = q_1 = C_1 V_1,$$

thus,

$$V_1 = \frac{q_1}{C_1} = \frac{q}{C_1} = \frac{1}{C_1}\frac{C_1(C_2 + C_3)}{(C_1 + C_2 + C_3)}V, \qquad \Rightarrow \qquad V_1 = \frac{(C_2 + C_3)}{(C_1 + C_2 + C_3)}V.$$

$$V_1 = \frac{(C_2 + C_3)}{(C_1 + C_2 + C_3)}V.$$

Let us consider the limit of V_1 when $C_3 \to \infty$.

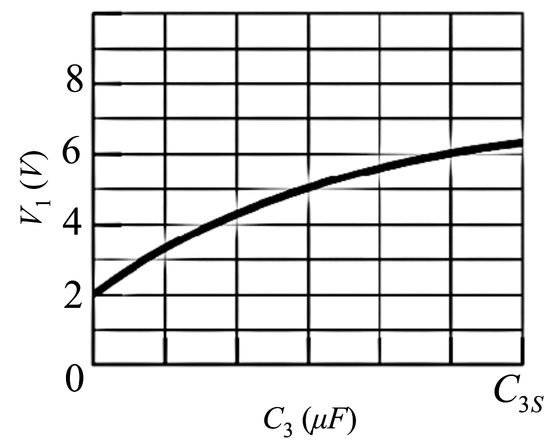
$$\lim_{C_3\to\infty} V_1 = \lim_{C_3\to\infty} \frac{(C_2+C_3)}{(C_1+C_2+C_3)} V = V \lim_{C_3\to\infty} \frac{\cancel{C_3}(1+C_2/C_3)}{\cancel{C_3}\left[1+(C_1+C_2)/C_3\right]} = V,$$

Using the $V_1 = 10\ V$ when $C_3 \to \infty$, we have that the potential difference across the battery is

$$V = 10V.$$

Let us consider the dependence of V_1 on C_3

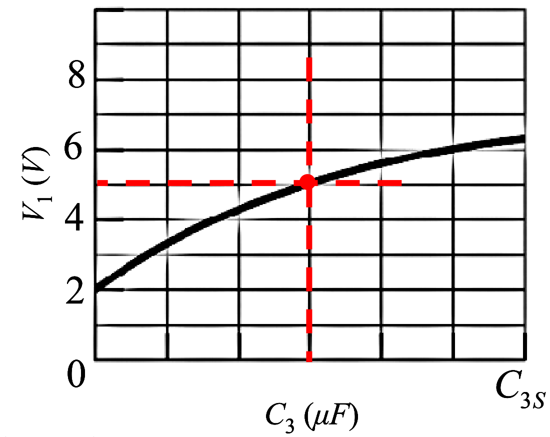
$$V_1 = \frac{(C_2+C_3)}{(C_1+C_2+C_3)} V.$$



When $C_3 = 0$, $V_1 = 2V$

$$2V = \frac{(C_2+0)}{(C_1+C_2+0)}(10V), \quad \Rightarrow \quad 1 = \frac{C_2}{(C_1+C_2)}5, \quad \Rightarrow \quad (C_1+C_2) = 5C_2,$$

$$C_1 + C_2 = 5C_2, \quad \Rightarrow \quad C_1 = 4C_2,$$



As the graph shows at C_3 = 6.0 μF, V_1 is exactly half of the battery voltage. Thus,

$$V_1 = \frac{(C_2 + C_3)}{(C_1 + C_2 + C_3)}V, \quad \Rightarrow$$

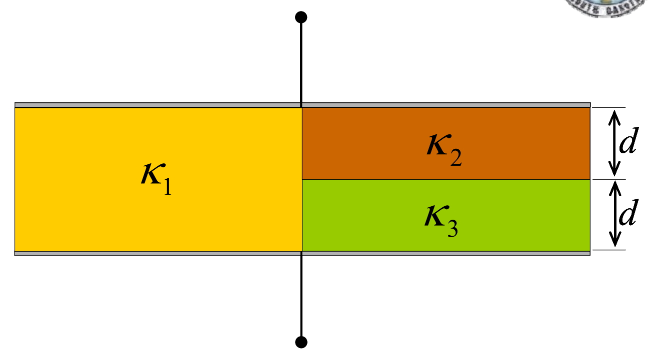
$$5V = \frac{(C_2 + 6.0\mu F)}{(4C_2 + C_2 + 6.0\mu F)}(10V),$$

$$1 = \frac{(C_2 + 6.0\mu F)}{(5C_2 + 6.0\mu F)}2, \quad \Rightarrow \quad (5C_2 + 6.0\mu F) = 2(C_2 + 6.0\mu F),$$

$$5C_2 + 6.0\mu F = 2C_2 + 12.0\mu F, \quad \Rightarrow \quad 3C_2 = 6.0\mu F, \quad C_2 = 3.0\mu F.$$

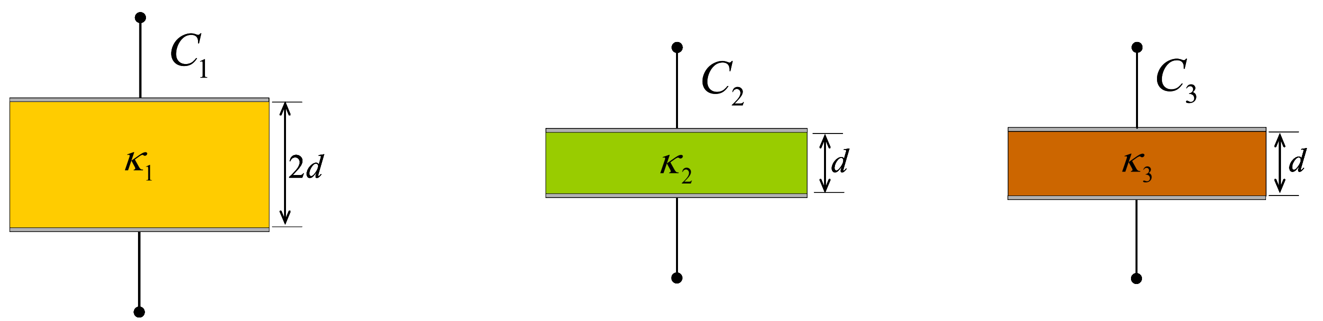
$$C_1 = 12.0\mu F.$$

***Problem*:** A parallel plate capacitor has the plate area $A = 10.5\ cm^2$ and plate separation $2d = 7.12\ mm$. *L*. Half of this capacitor is filled with dielectric material with dielectric constant $\kappa_1 = 21.0$ and another half is filled with two different dielectrics (see figure) with $\kappa_2 = 42.0$ and $\kappa_3 = 58.0$. Find the capacitance of the capacitor.



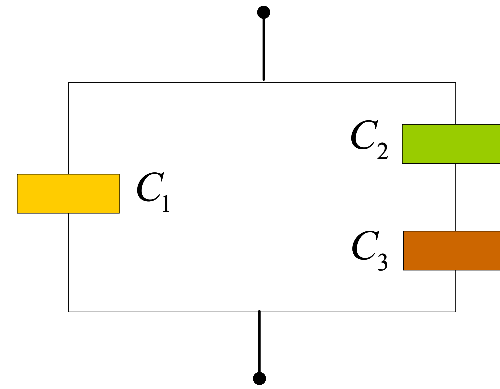
Solution

Let us consider this capacitor as a combination of three capacitors and figure out the type of their connections. The plate area of each capacitor is equal to $A/2$.



Capacitances of these capacitors are $C_1 = \kappa_1 \dfrac{\varepsilon_0 A}{4d}$, $C_2 = \kappa_2 \dfrac{\varepsilon_0 A}{2d}$, $C_3 = \kappa_3 \dfrac{\varepsilon_0 A}{2d}$.

The capacitors are connected as follows



$$C = C_1 + C_{23}, \qquad C_{23} = \frac{C_2 C_3}{(C_2 + C_3)},$$

$$C = C_1 + \frac{C_2 C_3}{(C_2 + C_3)} = \frac{C_1(C_2 + C_3) + C_2 C_3}{(C_2 + C_3)} = \frac{C_1 C_2 + C_1 C_3 + C_2 C_3}{(C_2 + C_3)},$$

$$C = \frac{C_1 C_2 + C_1 C_3 + C_2 C_3}{(C_2 + C_3)}.$$

$$C = \frac{\frac{\kappa_1 \varepsilon_0 A}{4d}\frac{\kappa_2 \varepsilon_0 A}{2d} + \frac{\kappa_1 \varepsilon_0 A}{4d}\frac{\kappa_3 \varepsilon_0 A}{2d} + \frac{\kappa_2 \varepsilon_0 A}{2d}\frac{\kappa_3 \varepsilon_0 A}{2d}}{\frac{\kappa_2 \varepsilon_0 A}{2d} + \frac{\kappa_3 \varepsilon_0 A}{2d}} = \frac{\frac{(\varepsilon_0 A)^2}{4d^2}\left[\frac{\kappa_1\kappa_2}{2} + \frac{\kappa_1\kappa_3}{2} + \kappa_2\kappa_3\right]}{\frac{\varepsilon_0 A}{2d}(\kappa_2 + \kappa_3)}$$

$$= \frac{\varepsilon_0 A\left[\frac{\kappa_1\kappa_2}{2} + \frac{\kappa_1\kappa_3}{2} + \kappa_2\kappa_3\right]}{2d(\kappa_2 + \kappa_3)} = \frac{\varepsilon_0 A}{2d}\frac{[\kappa_1\kappa_2 + \kappa_1\kappa_3 + 2\kappa_2\kappa_3]}{2(\kappa_2 + \kappa_3)},$$

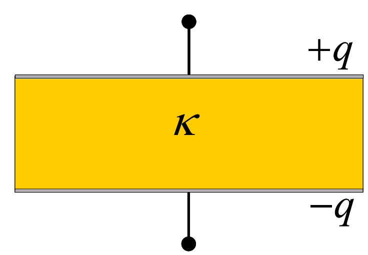
$$C = \frac{\varepsilon_0 A}{4d}\frac{(\kappa_1\kappa_2 + \kappa_1\kappa_3 + 2\kappa_2\kappa_3)}{(\kappa_2 + \kappa_3)}.$$

$$C = \frac{(8.85\cdot 10^{-12}\ C^2/N\cdot \cancel{m^2})(1.05\cdot \cancel{10^{-3}}\ \cancel{m^2})}{4(3.56\cdot \cancel{10^{-3}}\ m)} \frac{(21.0)\cdot(42.0)+(21.0)\cdot(58.0)+2(42.0)\cdot(58.0)}{(42.0)+(58.0)}$$

$$= \frac{8.85\cdot 1.05}{4\cdot 3.56} 10^{-12}\left(\frac{C^2}{N\cdot m}\right)\frac{882+1218+4872}{100} = 0.653\cdot 10^{-14}\left(\frac{C^2}{J}\right)(882+1218+4872)$$

$$= 0.653\cdot 10^{-14}\left(\frac{C}{V}\right)6.972\cdot 10^{3} = 4.55\cdot 10^{-11} F, \qquad C = 4.55\cdot 10^{-11} F.$$

***Problem*:** Two parallel plates of area 100 cm^2 are charged with equal magnitudes $q = 8.9\cdot 10^{-7}\ C$ but opposite signs of charge. The electric field within the dielectric material filling the space between plates is $1.4\cdot 10^{6}\ V/m$. Calculate (a) the dielectric constant of material, (b) the magnitude of the charge induced on the surfaces of dielectric.



Solution

(a) The dielectric constant of material is determined by $\kappa = E_0 / E$, where E_0 is the field in vacuum and E is the field in dielectric. To find the field in vacuum one can use the formula

$$E = \frac{\sigma}{\varepsilon_0}, \qquad \text{where} \qquad \sigma = q / A.$$

Thus,

$$\kappa = \frac{\sigma}{\varepsilon_0 E} = \frac{q}{\varepsilon_0 A E}.$$

$$\kappa = \frac{\left(8.9 \cdot 10^{-7}\, C\right)}{\left(8.85 \cdot 10^{-12}\, C^2 / N \cdot m^2\right)\left(10^{-2}\, m^2\right)\left(1.4 \cdot 10^{6}\, V / m\right)}$$

$$= \frac{8.9}{8.85 \cdot 1.4} \frac{10^{-7}}{10^{-12} \cdot 10^{-2} \cdot 10^{6}} \frac{C \cdot N \cdot m^2 \cdot m}{C^2 \cdot V \cdot m^2} = 0.72 \cdot 10 \frac{J}{C \cdot V} = 7.2 \frac{J \cdot C}{C \cdot J} = 7.2,$$

$$\kappa = 7.2.$$

The charge induced on the surface of dielectric is equal to

$$q' = q\left(1 - \frac{1}{\kappa}\right).$$

$$q' = \left(8.9\cdot 10^{-7}\ C\right)\left(1 - \frac{1}{7.2}\right) = 7.7\cdot 10^{-7}\ C,$$

$$q' = 7.7\cdot 10^{-7}\ C.$$