Note Title 2/8/2010

$(0,0)$ $(N-1,0)$

$$x(n_1, n_2)$$

$(0,N-1)$ $(N-1,N-1)$

$(0,0)$ $(M-1,0)$

$$h(n_1, n_2)$$

$(0,M-1)$ $(M-1,M-1)$

$(0,0)$ $(N+M-2,0)$

$$y(n_1, n_2)$$

$(0, N+M-2)$ $(N+M-2, N+M-2)$

* Each element of $y(n_1, n_2)$ will require
  M^2 multiplications, and
  $(M-1)^2$ additions

Elements near the border of the x matrix require fewer calculations (partial overlap), but usually $N \gg M$ (x is much bigger than h) so M^2 and $(M-1)^2$ are close enough

We have $(N+M-1) \times (N+M-1)$ elements in $y(n_1, n_2)$ so there are

$$\begin{cases} (N+M-1)^2 (M^2) & \text{multiplies, and} \\ (N+M-1)^2 (M-1)^2 & \text{adds} \end{cases}$$

Since $N \gg M$ though, this can be approximated as:

$$\left.\begin{aligned} \#\text{ multiplies} &\simeq N^2 M^2 \\ \#\text{ adds} &\simeq N^2 M^2 \end{aligned}\right\} \begin{aligned} &N \simeq 1000 \quad \text{image} \\ &M \simeq 25 \quad \text{filter coefficients} \end{aligned}$$

For Direct Linear Convolution from the formula

* Assume however that $h(n_1, n_2)$ is seperable; that is

$$h(n_1, n_2) = h(n_1)\, h(n_2)$$

Then $$y(n_1, n_2) = \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} h_1(n_1 - m_1)\, h_2(n_2 - m_2)\, x(m_1, m_2)$$

$$= \sum_{m_1=-\infty}^{+\infty} h_1(n_1 - m_1) \cdot \underbrace{\left[\sum_{m_2=-\infty}^{\infty} h_2(n_2 - m_2)\, x(m_1, m_2) \right]}_{\text{Compute all this for each (fixed) value of } m_1}$$

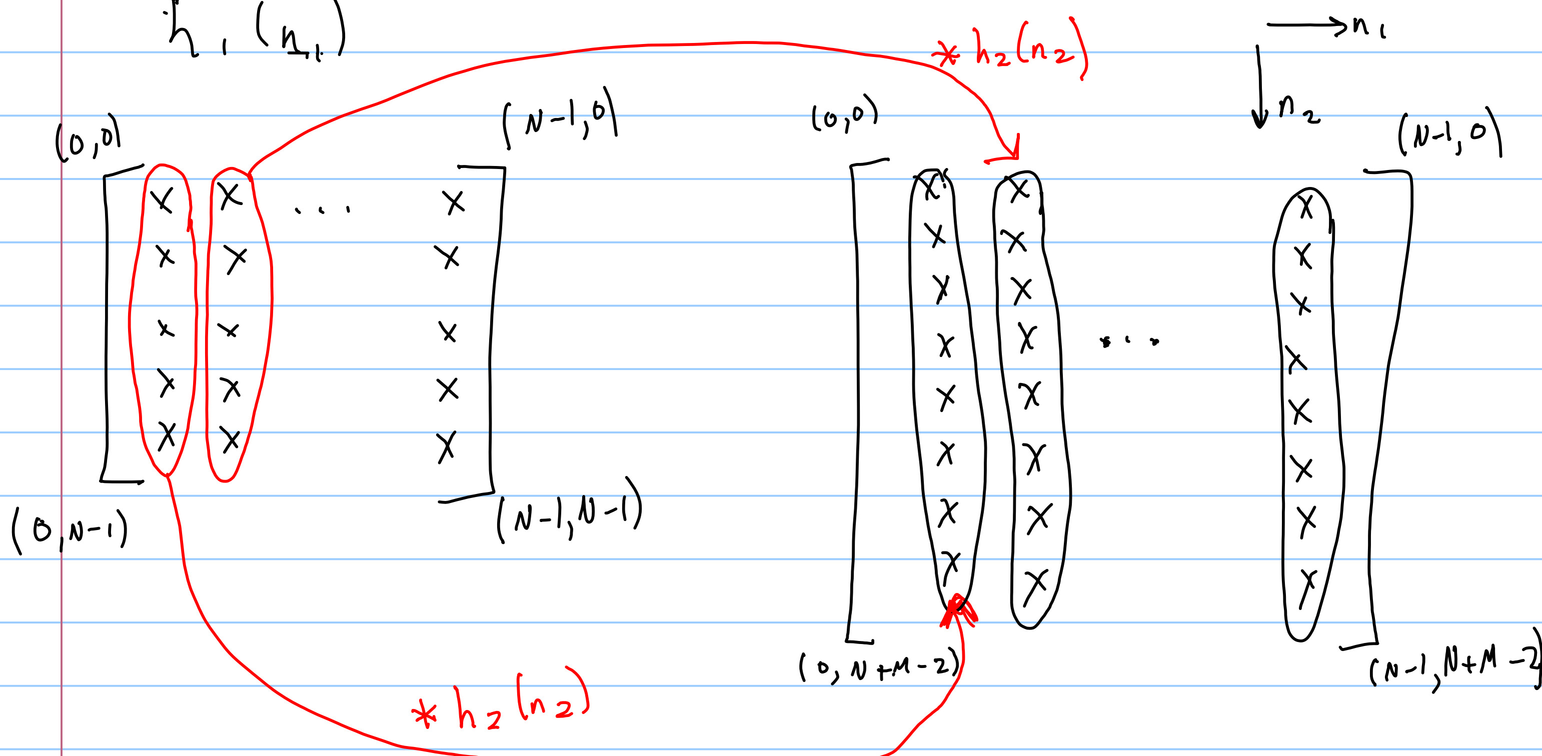
$$\sum_{m_2=-\infty}^{+\infty} h_2(n_2-m_2)\,x(m_1,m_2) \Rightarrow f(m_1,n_2)\Big|_{\text{fixed } m_1} = h_2(n_2) * x(m_1,n_2)_{\text{fixed } m_1}$$

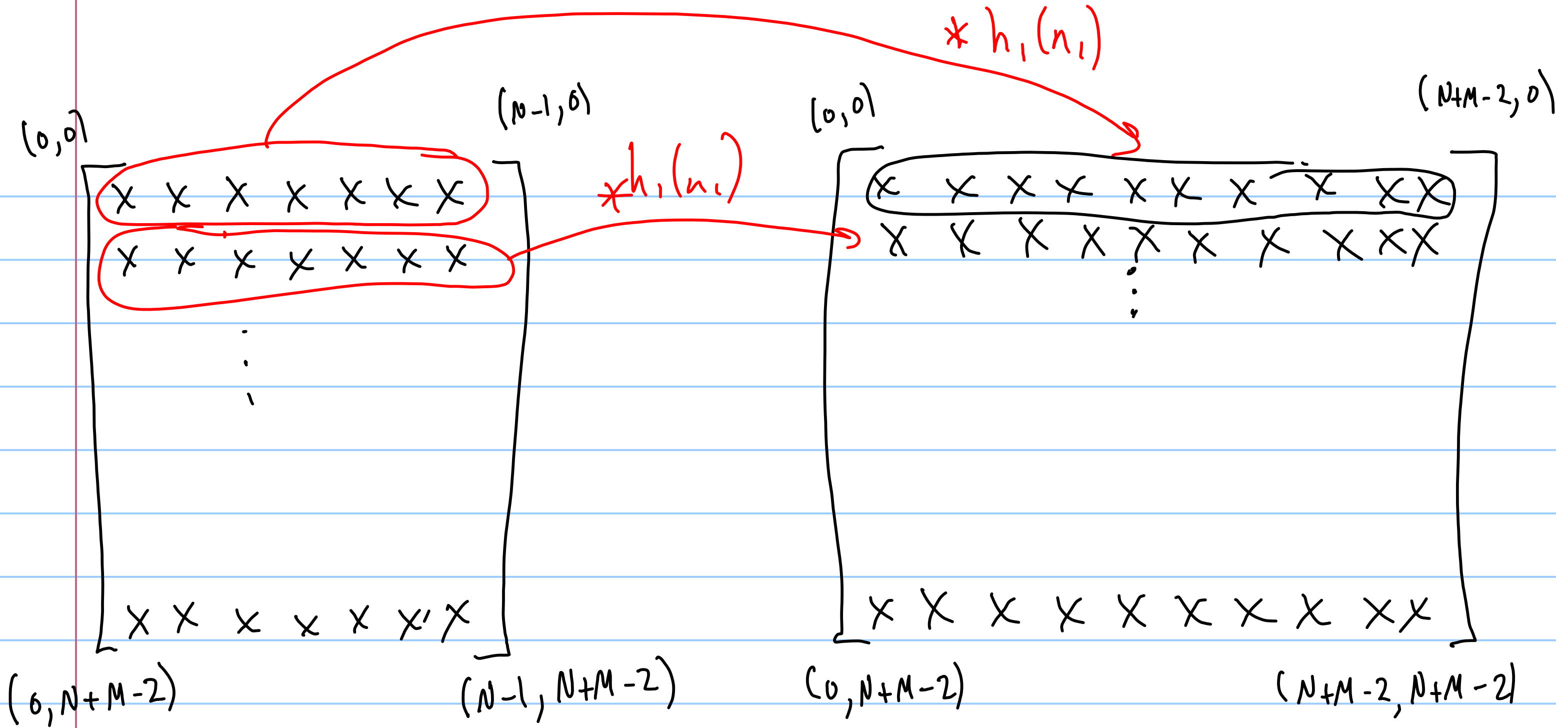
* This is a 1-D convolution of a column of x with $h_2(n_2)$

Then $y(n_1,n_2) = \sum_{m_1=-\infty}^{+\infty} h_1(n_1-m_1)\,f(m_1,n_2)$

But if we fix n_2 constant, that is, pick a particular <u>row</u> of $f(m_1,n_2)$ then this is the definition of the 1-D convolution of the <u>rows</u> of $f(m_1,n_2)$ with $h_1(n_1)$

* So we can do 2-D separable convolution by first doing 1-D convolution of the columns with $h_2(n_2)$ and then do 1-D convolution across the rows of the result with $h_1(n_1)$





$$h(n_1,n_2) = h_1(n_1)\,h_2(n_2)$$

$$h_1 = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix} \qquad h_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$x(n_1,n_2) = \begin{bmatrix} 3 & 4 & 0 & -2 \\ -2 & 6 & 4 & -3 \\ 0 & 5 & 2 & -4 \\ 5 & -2 & 3 & 0 \end{bmatrix}$$

Convolve h_2 vertically with each column of $x(n_1,n_2)$

	1		1		1		1
	2 1		2 1		2 1		2
3	1 2 1	4	1 2 1	0	1 2 1	−2	1
−2	1 2 1	6	1 2 1	4	1 2 1	−3	
0	1 2 1	5	1 2 1	2	1 2 1	−4	
5	1 2 1	−2	1 2 1	3	1 2 1	0	
	1 2		1 2		1 2		
	1		1		1		

$$\begin{bmatrix} 3 & 4 & 0 & -2 \\ 4 & 14 & 4 & -7 \\ -1 & 21 & 10 & -12 \\ 3 & 14 & 11 & -11 \\ 10 & 1 & 8 & -4 \\ 5 & -2 & 3 & 0 \end{bmatrix}$$

Now take this and convolve h_1 with each row

$$3 \quad 4 \quad 0 \quad -2 \quad = \quad -3 \quad 2 \quad 5 \quad -2 \quad -4 \quad 2$$

−1 2 −1
−1 2 −1
−1 2 −1
−1 2 −1
−1 2 −1
−1 2 −1

$$4 \quad 14 \quad 4 \quad -7 = -4 \quad -6 \quad 20 \quad 1 \quad -18 \quad 7$$

$$-1 \quad 2 \quad -1$$

$$-1 \quad 21 \quad 10 \quad -12 = -23 \quad 33 \quad 11 \quad -34 \quad 12$$

$$-1 \quad 2 \quad -1$$

$$y(n_1, n_2) = x(n_1, n_2) * h(n_1, n_2)$$

$$= \begin{bmatrix} -3 & 2 & 5 & -2 & -4 & 2 \\ -4 & -6 & 20 & 1 & -18 & 7 \\ 1 & -23 & 33 & 11 & -34 & 12 \\ -3 & -8 & 14 & 19 & -33 & 11 \\ -10 & 19 & -16 & 19 & -16 & 4 \\ -5 & 12 & -12 & 8 & -3 & 0 \end{bmatrix}$$

The length of the column convolutions are $N+M-1$ each of which requires approximately M multiplies and $M-1$ additions

So the first stage needs per column

$$\simeq \quad (N+M-1)(M) \quad \text{multiplies}$$
$$(N+M-1)(M-1) \quad \text{adds}$$

There are N column so total for the first stage is

$$(N+M-1)(M)(N) \quad \text{multiplies}$$
$$(N+M-1)(M-1)(N) \quad \text{adds}$$

$$\text{for } N \gg M \implies \begin{array}{l} N^2 M \\ N^2 M \end{array}$$

For row convolutions with $h_1(n_1)$
so each row is N long convolved with M-length h_1
Result is $N+M-1$ long

So each row convolution requires
$$(N+M-1)(M) \quad \text{mults}$$
$$(N+M-1)(M-1) \quad \text{adds}$$

There are now $N+M-1$ total rows (not N rows)

So total row convolution computations are

$$(N+M-1)(M)(N+M-1) \quad \text{mult}$$
$$(N+M-1)(M-1)(N+M-1) \quad \text{adds}$$

for $N \gg M$ this is $\simeq$

$$N^2 M \quad \text{mults}$$
$$N^2 M \quad \text{adds}$$

Total 2-D separable convolution

$$2N^2 M \quad \text{mults}$$
$$2N^2 M \quad \text{adds}$$

Direct 2D convolution

$$N^2 M^2 \quad \text{multiplies}$$
$$N^2 M^2 \quad \text{adds}$$

$$\frac{N^2 M^2}{2N^2 M} = \frac{M}{2}$$

2-D separable version is factor of $\frac{M}{2}$ faster