Math 424 Exam I 2/19/10 Solutions

① Consider $\lim_{x\to 0} \dfrac{\frac{1}{x}}{e^{1/x^2}}$, which has the indeterminate form $\frac{\infty}{\infty}$.

It meets the hypotheses of Thm 5.5.6, and

$$\lim_{x\to 0^+} \frac{f'}{g'} = \lim_{x\to 0^+} \frac{-x^{-2}}{-2x^{-3}e^{1/x^2}} = \lim_{x\to 0^+} \frac{x}{2e^{1/x^2}} = 0.$$

Next, consider $\lim_{x\to 0^-} \dfrac{e^{-1/x^2}}{x}$, and let $y=-x$. Then

$$\lim_{x\to 0^-} \frac{e^{-1/x^2}}{x} = \lim_{y\to 0^+} \frac{e^{-1/y^2}}{-y} = -\lim_{y\to 0^+} \frac{e^{-1/y^2}}{y} = -0 = 0.$$

Therefore $\lim_{x\to 0} \dfrac{e^{-1/x^2}}{x} = 0.$

EC Show $\lim_{x\to 0} \dfrac{e^{-1/x^2}}{x^n} = 0.$

The result is established above for $n=1$. Suppose it is true for n. Then

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^{n+1}} = \lim_{x\to 0^+} \frac{\frac{1}{x^{n+1}}}{e^{1/x^2}} = \lim_{x\to 0^+} \frac{-(n+1)x^{-(n+2)}}{-2x^{-3}e^{1/x^2}}$$

$$= \frac{n+1}{2} \lim_{x\to 0^+} \frac{x\,e^{-1/x^2}}{x^n} = \frac{n+1}{2} \lim_{x\to 0^+} x \lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = \frac{n+1}{2}\cdot 0\cdot 0 = 0.$$

As before, set $y=-x$, so that

$$\lim_{x\to 0^-} \frac{e^{-1/x^2}}{x^{n+1}} = (-1)^{n+1} \lim_{y\to 0^+} \frac{e^{-1/y^2}}{y^{n+1}} = (-1)^{n+1}\cdot 0 = 0.$$

Hence $\lim_{x\to 0} \dfrac{e^{-1/x^2}}{x^{n+1}} = 0$, and the induction is complete.

② a) Let $\{A_\alpha\}_{\alpha \in G}$ be any open cover of E. For some $\alpha_0 \in G$, $1 \in A_{\alpha_0}$. Since A_{α_0} is open, it contains an open interval containing 1; that is, $1 \in (1-\delta, 1+\delta) \subseteq A_{\alpha_0}$ for some $\delta > 0$. Since $\lim_{n\to\infty} \frac{n}{n+1} = 1$, there is an n^* such that $\frac{n}{n+1} \in (1-\delta, 1+\delta)$ for all $n > n^*$. Now choose $\alpha_1 \in G$ so that $\frac{1}{2} \in A_{\alpha_1}$, $\alpha_2 \in G$ so that $\frac{2}{3} \in A_{\alpha_2}$, $\ldots$, $\alpha_{n^*} \in G$ so that $\frac{n^*}{n^*+1} \in A_{\alpha_{n^*}}$. (This is possible since $\{A_\alpha\}_{\alpha\in G}$ covers E.) The collection $\{A_{\alpha_0}, A_{\alpha_1}, \ldots, A_{\alpha_{n^*}}\}$ is the required finite subcover.

b) The collection $\{(\frac{1}{n}, 1) \mid n \in \mathbb{N}\}$ is an open cover of $(0, \frac{1}{2}]$. Suppose a finite subcover $S = \{(\frac{1}{n_1}, 1), \ldots, (\frac{1}{n_k}, 1)\}$ exists. Let $n^* = \max\{n_i \mid i = 1, 2, \ldots, k\}$. Then $\frac{1}{n^*+1} \in (0, \frac{1}{2}]$, yet $\frac{1}{n^*+1} < \frac{1}{n^*} \le \frac{1}{n_i}$ (for $i = 1, \ldots, k$). Therefore $\frac{1}{n^*+1} \notin (\frac{1}{n_i}, 1)$ for $i = 1, \ldots, k$, so S does not cover $(0, \frac{1}{2}]$, a contradiction! Thus no finite subcover exists.

c) Every irrational number in $[0,1]$ is an accumulation point of $\mathbb{Q} \cap [0,1]$, yet is not in $\mathbb{Q} \cap [0,1]$. Thus $\mathbb{Q} \cap [0,1]$ is not closed. By the Heine Borel theorem, it is therefore not compact.

③ a) $f(x) = (x+1)^{1/2}$ $\quad f'(x) = \frac{1}{2}(x+1)^{-1/2}$ $\quad f''(x) = -\frac{1}{4}(x+1)^{-3/2}$

$f(0) = 1$ $\quad f'(0) = \frac{1}{2}$ $\quad f''(0) = -\frac{1}{4}$

$$p_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)(x-0)^2}{2!}$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

b) $f(x) = p_2(x) + R_2(x)$, where $R_2(x) = \frac{f^3(c)\,x^3}{3!} = \frac{\frac{3}{8}(c+1)^{-5/2}x^3}{6} = \frac{x^3}{16(c+1)^{5/2}}$

for some $c \in (0,x)$. Since $c>0$ and $x>0$, we know

$0 < R_2(x) = f(x) - p_2(x)$, therefore $p_2(x) < f(x)$.

④ Since e^{-t^2} is continuous on $\mathbb{R}$, theorem 6.4.4 tells us that $F(x) = \int_0^x e^{-t^2}\,dt$ is

1) differentiable on $[0,\infty)$, therefore
2) continuous on $[0,\infty)$.

Also by theorem 6.4.4, we have

3) $F'(x) = e^{-x^2}$ for $x \in [0,\infty)$.

Since $e^{-x^2} > 0$ (always), item 3 tells us that F is

4) strictly increasing; therefore
5) one-to-one; therefore
6) F has an inverse.

From theorem 6.3.2, (or since $F(0) = 0$ and F is increasing) we have

7) F is non-negative.

It is also true that F is bounded, but not trivial to show.

5 a) The key observation is that the partial sums satisfy

$$s_1 = a_1 - a_2$$

$$s_2 = a_1 - a_2 + a_2 - a_3 = a_1 - a_3$$

$$\vdots$$

$$s_n = a_1 - a_{n+1}$$

If the series converges, then $\lim_{n\to\infty} s_n = L$ (finite), so

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (a_1 - s_n) = a_1 - L$, so $\{a_n\}$ converges.

Likewise, if $\{a_n\}$ converges, say to A (finite), then

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (a_1 - a_{n+1}) = a_1 - A$, so the series converges.

b) The preceding work shows that the sum is $a_1 - A$.