Discrete Cosine Transform

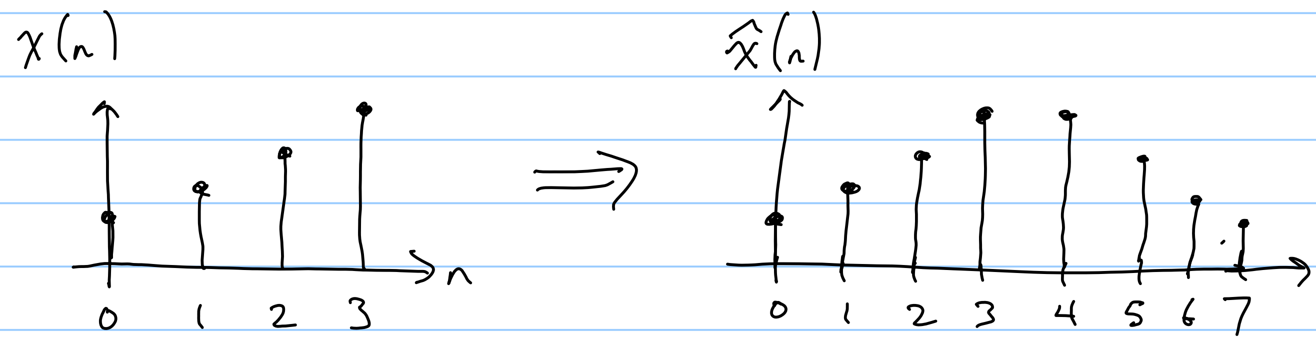
Note Title 4/23/2010

(1) DCT coefficients are purely real.
(2) DCT very good at energy compaction
(3) DCT can be computed on the same order as FFT.

We can define or create the DCT in this way:

* Take an N-length sequence and make it symmetrical by:

$$\hat{x}(n) = \begin{cases} x(n), & 0 \le n \le N-1 \\ x(2N-1-n), & N \le n \le 2N-1 \end{cases}$$



$\hat{x}(n)$ is of length $2N$ and is symmetrical

* Take the 2N-point DFT of $\hat{x}(n)$

$$\hat{X}(k) = \sum_{n=0}^{N-1} x(n) W_{2N}^{kn} + \sum_{n=N}^{2N-1} x(2N-1-n) W_{2N}^{kn}$$

For the second half of the sequence (where $N \le n \le 2N-1$) let $p = 2N-1-n$ and change variables

$$\hat{X}(k) = \sum_{n=0}^{N-1} x(n) W_{2N}^{kn} + \sum_{p=N-1}^{0} x(p) W_{2N}^{k(2N-1-p)}$$

It doesn't matter which order we add the summation terms together, so reverse the order for the second summation.

$$\hat{x}(k) = \sum_{n=0}^{N-1} x(n) W_{2N}^{kn} + \sum_{p=0}^{N-1} x(p) W_{2N}^{k(2N-1-p)}$$

Now recall what our twiddle factors are

$$W_N = e^{-j\frac{2\pi}{N}}$$

So $$W_{2N}^{kn} = \left(e^{-j\frac{2\pi}{2N}}\right)^{kn} = e^{-j\frac{2\pi kn}{2N}}$$

and $$W_{2N}^{k(2N-1-p)} = e^{-j\frac{2\pi}{2N}k(2N-1-p)}$$

$$= \left(e^{-j\frac{2\pi}{2N}k2N}\right)\left(e^{-j\frac{2\pi}{2N}(-k)}\right)\left(e^{-j\frac{2\pi}{2N}\{-kp\}}\right)$$

Thus $$W_{2N}^{k(2N-1-p)} = W_{2N}^{2kN} W_{2N}^{-k} W_{2N}^{-kp}$$

but $$W_{2N}^{2kN} = e^{-j\frac{2\pi k2N}{2N}} = 1$$

So $$\hat{X}(k) = \sum_{n=0}^{N-1} x(n) W_{2N}^{kn} + \sum_{p=0}^{N-1} x(p) W_{2N}^{-k} W_{2N}^{-kp}$$

Replace p with n (remember the second sequence is a mirror of the first so $x(p)$ from $p = N-1$ to 0 is same as $x(n)$ for $n = 0$ to $N-1$)

Then $$\hat{X}(k) = \sum_{n=0}^{N-1} x(n) W_{2N}^{kn} + \sum_{n=0}^{N-1} x(n) W_{2N}^{-(n+1)k}$$

$$\hat{X}(k) = W_{2N}^{-k/2} \sum_{n=0}^{N-1} x(n) \left[W_{2N}^{k(n+\frac{1}{2})} + W_{2N}^{-k(n+\frac{1}{2})} \right]$$

But $W_{2N}^{k(n+\frac{1}{2})}$ is $e^{-j\frac{2\pi}{2N}k(n+\frac{1}{2})}$

So we have $\left[e^{-j\frac{\pi k}{N}(n+\frac{1}{2})} + e^{+j\frac{\pi k}{N}(n+\frac{1}{2})}\right]$

and $\cos(\alpha) = \dfrac{e^{-j\alpha}+e^{+j\alpha}}{2}$

and therefore $W_{2N}^{k(n+\frac{1}{2})} + W_{2N}^{-k(n+\frac{1}{2})} = 2\cos\left(\frac{\pi k}{N}(n+\frac{1}{2})\right)$

Finally $\hat{X}(k) = W_{2N}^{-k/2} \sum_{n=0}^{N-1} 2x(n)\cos\left(\frac{\pi k}{N}(n+\frac{1}{2})\right)$

for $0 \le k \le 2N-1$

Let the DCT be defined as

$$\begin{cases} W_{2N}^{k/2}\hat{X}(k), & \text{for } \emptyset \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

This makes the DCT purely real

The DCT of $x(n)$ is then:

$$C_x(k) = \sum_{n=0}^{N-1} 2x(n)\cos\frac{(2n+1)\pi k}{2}$$

Usually this is scaled so it is unitary and thus

$$C_x(k) = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos \frac{(2n+1)k\pi}{2N}$$

where $\alpha(0) = \frac{1}{\sqrt{N}}$, $\alpha(k) = \sqrt{\frac{2}{N}}$ for $1 \le k \le N-1$

The inverse unitary DCT is

$$x(n) = \sum_{k=0}^{N-1} \alpha(k)\, C_x(k) \cos \frac{(2n+1)\pi k}{2N}$$

We can write this in a matrix form like we did for the DFT

$$\vec{C} = \begin{array}{c} \left[\begin{array}{cccc} \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \cdots & \frac{1}{\sqrt{N}} \\ \sqrt{\frac{2}{N}} \cos \frac{\pi}{2N} & \sqrt{\frac{2}{N}} \cos \frac{3\pi}{2N} & \cdots & \sqrt{\frac{2}{N}} \cos \frac{(2(N-1)+1)\pi}{2N} \\ \vdots & & \ddots & \\ \sqrt{\frac{2}{N}} \cos \frac{(N-1)\pi}{2N} & \sqrt{\frac{2}{N}} \cos \frac{3(N-1)\pi}{2N} & \cdots & \sqrt{\frac{2}{N}} \cos \frac{(2(N-1)+1)(N-1)\pi}{2N} \end{array}\right] \begin{array}{l} k=0 \text{ row} \\ k=1 \text{ row} \\ \\ k=N-1 \text{ row} \end{array} \\ \begin{array}{cccc} n=0 \text{ column} & n=1 \text{ column} & & n=N-1 \text{ column} \end{array} \end{array}$$

And so $\vec{V} = \vec{C} \vec{x}$

Properties of the DCT

(1) DCT is real and orthonormal

$$\vec{C} = \vec{C}^* \quad \text{and} \quad \vec{C}\,\vec{C}^T = \vec{I}$$

(2) DCT is a fast transform

The DCT of an N-vector can be computed in $O(N \log_2 N)$ operations

Split the samples of $x(n)$ into even and odd parts

Let $\tilde{x}(n) = x(2n)$

$$\tilde{x}(N-n-1) = x(2n+1) \quad \text{for} \quad 0 \le n \le \frac{N}{2} - 1$$

So the DCT of the sequence $x(n)$ is

$$C_x(k) = \alpha(k)\left[\sum_{n=0}^{\frac{N}{2}-1} x(2n) \cos\frac{\pi(4n+1)k}{2N} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) \cos\frac{\pi(4n+3)k}{2N}\right]$$

$$= \alpha(k)\left[\sum_{n=0}^{\frac{N}{2}-1} \tilde{x}(n) \cos\frac{\pi(4n+1)k}{2N} + \sum_{n=0}^{\frac{N}{2}} \tilde{x}(N-n-1) \cos\frac{\pi(4n+3)k}{2N}\right]$$

Now Let $m = N-n-1$ to give (for the 2nd half)

$$C_x(k) = \alpha(k)\left[\sum_{n=0}^{\frac{N}{2}-1} \tilde{x}(n) \cos\frac{\pi(4n+1)k}{2N} + \sum_{m=N-1}^{N/2} \tilde{x}(m) \cos\left(\frac{\pi k(4[N-m-1]+3)}{2N}\right)\right]$$

$$C_x(k) = \alpha(k)\left[\sum_{n=0}^{\frac{N}{2}-1} \tilde{x}(n) \cos\frac{\pi(4n+1)k}{2N} + \sum_{m=N-1}^{N/2} \tilde{x}(m) \cos\left(\frac{\pi k[4N-4m-4+3]}{2N}\right)\right]$$

$$C_x(k) = \alpha(k)\left[\sum_{n=0}^{\frac{N}{2}-1} \tilde{x}(n)\cos\frac{\pi(4n+1)k}{2N} + \sum_{m=N-1}^{N/2} \tilde{x}(m)\cos\left(\frac{4N\pi k - \pi k[4m+1]}{2N}\right)\right]$$

$$C_x(k) = \alpha(k)\left[\sum_{n=0}^{\frac{N}{2}-1} \tilde{x}(n)\cos\frac{\pi(4n+1)k}{2N} + \sum_{m=N-1}^{N/2} \tilde{x}(m)\cos\left(2\pi k - \frac{\pi k[4m+1]}{2N}\right)\right]$$

$$C_x(k) = \alpha(k)\sum_{n=0}^{N-1} \tilde{x}(n)\cos\frac{\pi(4n+1)k}{2N}$$

$$= \alpha(k)\sum_{n=0}^{N-1} \tilde{x}(n)\, Re\left\{ e^{-j\left(\frac{2\pi nk}{N} + \frac{\pi k}{2N}\right)}\right\}$$

$$= Re\left\{\alpha(k) e^{-j\frac{\pi k}{2N}} \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2\pi nk}{N}}\right\}$$

$$= Re\left\{\alpha(k) e^{-j\frac{\pi k}{2N}}\, DFT\left[\tilde{x}(n)\right]\right\}$$

(3) The DCT has excellent energy compaction for highly correlated data

For a signal $x(n)$ that generates a smooth extension for $x(n) + x(N+n-1)$ the energy is packed into a small number of coefficients

As this is the case for most images, the 2-D DCT is commonly used for image compression.

$\ast$ 2-D unitary DCT of an $N \times N$ sequence $x(n_1, n_2)$ is defined by

$$C_x(k_1, k_2) = \alpha(k_1)\alpha(k_2) \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} x(n_1, n_2) \cos\frac{(2n_1+1)\pi k_1}{2N} \cos\frac{(2n_2+1)\pi k_2}{2N}$$

This is separable. Therefore, the 2-D DCT can be performed in terms of 1-D DCT's using row-column decomposition.

The inverse unitary DCT is given by

$$x(n_1, n_2) = \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} \alpha(k_1)\alpha(k_2) C_x(k_1, k_2) \cos\frac{(2n_1+1)\pi k}{2N} \cos\frac{(2n_2+1)\pi k}{2N}$$

For a given input matrix X, its 2-D unitary DCT is

$$\vec{V} = \vec{C}\,\vec{X}\,\vec{C}^T$$

and the inverse 2-D DCT is

$$\vec{X} = \vec{C}^T\,\vec{V}\,\vec{C}$$