Math 424 Exam II April 12, 2010

(8.4.1b)

Let $f_k(x) = x^k e^{-kx}$, with $x \in [0, \infty)$. Differentiate to obtain $f_k'(x) = k x^{k-1} e^{-kx}(1-x)$. Observe that $f_k(x)$ increases on $[0,1]$ and decreases on $[1,\infty)$, so it has a global maximum at $x = 1$. The maximum value is $e^{-k}$. Now apply the Weierstrass M test, with $M_k = e^{-k}$. We have $|f_k(x)| \le M_k$ for all $x$ in the domain, and $\sum M_k$ is a convergent geometric series, thus $\sum f_k(x)$ converges uniformly on $[0,\infty)$.

(2) For $(x,y) \neq (0,0)$, the computation is straightforward:

$$f_x(x,y) = \frac{2}{3}(x^2+y^2)^{-1/3}(2x) = \frac{4x}{3(x^2+y^2)^{1/3}}.$$

At $(0,0)$, apply the definition of the partial derivative:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^{4/3}}{h} = \lim_{h\to 0} h^{1/3} = 0.$$

(3) a) Both Bob and Fred meet the hypotheses of theorem 7.4.2 (the alternating series test); therefore both converge.

b) Bob is <u>conditionally</u> convergent: $\sum \left| \frac{(-1)^{k+1}}{\sqrt{k}} \right| = \sum \frac{1}{\sqrt{k}}$ which is a divergent p-series (example 7.2.3). Thus Bob is convergent, but not absolutely convergent, which by definition makes Bob conditionally convergent. Now apply Riemann's rearrangement theorem (7.4.16): there exists a rearrangement that converges to the same sum as Fred.

c) Fred is <u>absolutely</u> convergent, since $\sum \left| \frac{(-1)^{k+1}}{k\sqrt{k}} \right| = \sum \frac{1}{k^{3/2}}$, a convergent p-series. Theorem 7.4.15 guarantees that every rearrangement of Fred converges with the same sum. So Fred <u>cannot</u> be rearranged to have the same sum as Bob.

④ a) Let $\epsilon > 0$ be given. If $x = 0$, then

$|h_n(x) - h(x)| = |h_n(0) - h(0)| = |0 - 0| = 0 < \epsilon$ for all $n \in \mathbb{N}$.

Next, fix $x \in (0,1]$. Choose $n^* > \frac{1}{x}$, which makes $\frac{1}{n^*} < x$. If $n > n^*$ then $\frac{1}{n} < \frac{1}{n^*} < x$, so

$|h_n(x) - h(x)| = \left| \frac{1}{x} - \frac{1}{x} \right| = 0 < \epsilon$. Therefore $\{h_n(x)\}$ converges pointwise on $[0,1]$ to $h(x)$.

b) Since $h_n(x) = n^2 x$ is a continuous (polynomial) function on $[0, \frac{1}{n})$ and $h_n(x) = \frac{1}{x}$ is a continuous rational function (quotient of continuous polynomial functions with nonzero denominator) on $(\frac{1}{n}, 1]$, we need only consider continuity at $x = \frac{1}{n}$. But

$\lim_{x \to \frac{1}{n}^-} h_n(x) = n^2 \cdot \frac{1}{n} = n$ and $\lim_{x \to \frac{1}{n}^+} h_n(x) = \frac{1}{\frac{1}{n}} = n$,

therefore $\lim_{x \to \frac{1}{n}} h_n(x) = n$. Also $h_n\left(\frac{1}{n}\right) = \frac{1}{\frac{1}{n}} = n$,

so $h_n(x)$ is continuous at $x = \frac{1}{n}$.

c) Suppose $\{h_n(x)\}$ converges uniformly to $h(x)$ on $[0,1]$. From part b, each $h_n(x)$ is continuous on $[0,1]$, therefore $h(x)$ must be continuous on $[0,1]$ (theorem 8.3.1). But this is clearly false: $h(x)$ is <u>not</u> continuous at $x = 0$. Therefore the convergence of $\{h_n(x)\}$ cannot be uniform.

(5) a) $\lim_{(x,y)\to(a,b)} f(x,y) \neq L$ means that there exists $\epsilon > 0$ such that for any $\delta > 0$, there is a point $(x,y)$ in $D$ with $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, yet $|f(x,y) - L| > \epsilon$.

b) Suppose $L = 0$. Choose $\epsilon = \frac{1}{3}$, and let $\delta > 0$ be given. If $\delta > 1$, choose $x = \frac{1}{2}$ and $y = \frac{1}{4}$. Then $0 < \sqrt{(\frac{1}{2}-0)^2+(\frac{1}{4}-0)^2} = \frac{\sqrt{5}}{4} < 1 < \delta$, yet $|f(x,y) - L| = |\frac{1}{2} - 0| = \frac{1}{2} > \frac{1}{3} = \epsilon$. If $\delta \le 1$, choose $x = \frac{\delta}{2}$ and $y = \frac{\delta^2}{4}$. Then $0 < \sqrt{\left(\frac{\delta}{2}\right)^2+\left(\frac{\delta^2}{4}\right)^2} = \frac{\delta}{2}\sqrt{1+\frac{\delta^2}{4}} \le \frac{\delta}{2}\frac{\sqrt{5}}{2} < \delta$, but $|f(x,y) - L| = \frac{1}{2} > \frac{1}{3} = \epsilon$. Therefore $\lim_{(x,y)\to(0,0)} f(x,y) \neq L$.

Suppose $L \neq 0$. Choose $\epsilon = \frac{1}{2}|L|$, and let $\delta > 0$ be given. Choose $x = 0$, $y = \frac{\delta}{2}$. Then $0 < \sqrt{(0-0)^2+(\frac{\delta}{2}-0)^2} = \frac{\delta}{2} < \delta$, but $|f(x,y) - L| = |0 - L| = |L| > \frac{1}{2}|L| = \epsilon$. Therefore $\lim_{(x,y)\to(0,0)} f(x,y) \neq L$.

Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.